THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW7 Solution

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1. (P.246 Q4)

   Case 1: $0 \le x < 1$: Since $\lim_{n\to\infty} x^n = 0$, we have the following:

   $$\lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0$$

   Case 2: $x = 1$: Then

   $$\lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{1}{1+1} = \frac{1}{2}$$

   Case 3: $1 < x < +\infty$: Since $\lim_{n\to\infty} \frac{1}{x^n} = 0$, we have the following:

   $$\lim_{n\to\infty} \frac{x^n}{1+x^n} = \lim_{n\to\infty} \frac{1}{\frac{1}{x^n}+1} = \frac{1}{0+1} = 1$$

2. (P.246 Q8)

   We claim that $\lim_{n\to\infty} xe^{-nx} = 0$ for all $x \ge 0$:

   Let $\epsilon > 0$ be given, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then by the inequality in Example 6.2.10 of the textbook, $e^x \ge 1 + x$ for all $x \in \mathbb{R}$. Therefore, for all $n \ge N$, $x \ge 0$

   $$\begin{aligned} e^{nx} &\ge 1 + nx \\ &> Nx \\ &> \frac{x}{\epsilon} \end{aligned}$$

   which implies $xe^{-nx} < \epsilon$ for all $n \ge N$. Therefore, for all $x \ge 0$, $\lim_{n\to\infty} xe^{-nx} = 0$.

3. (P.247 Q14)

   (i) Fix $0 < b < 1$, then by (4), for all $x \in [0, b]$, $\lim_{n\to\infty} \frac{x^n}{1+x^n} = 0$. We claim the convergence is uniform in $[0, b]$:

Given $\epsilon > 0$, since $\lim_{n\to\infty} b^n = 0$, there exists $N \in \mathbb{N}$ such that $b^N < \epsilon$. Then for all $n \geq N$, $x \in [0, b]$,

$$\begin{aligned} \left|\frac{x^n}{1+x^n}\right| &\leq \frac{b^n}{1+0} \\ &\leq b^N \\ &< \epsilon \end{aligned}$$

Therefore, the convergence is uniform in $[0, b]$.

(ii) We claim that the convergence is not uniform in $[0, 1]$: By Q4, if the convergence were uniform, the uniform limit function would be given by

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ \frac{1}{2} & x = 1 \end{cases}$$

We use Lemma 8.15 of the textbook to show that $f_n(x) = \dfrac{x^n}{1+x^n}$ does not converge to f: Since $\lim_{n\to\infty}(1 - \frac{1}{n})^n = e^{-1} > \frac{1}{3}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $(1 - \frac{1}{n})^n > \frac{1}{3}$.
Choose $\epsilon_0 = \frac{1}{4}$, $n_k = k + N$, $x_k = 1 - \dfrac{1}{k+N}$. Then

$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= \frac{(1 - \frac{1}{k+N})^{k+N}}{1 + (1 - \frac{1}{k+N})^{k+N}} \\ &= \frac{1}{[(1 - \frac{1}{k+N})^{k+N}]^{-1} + 1} \\ &> \frac{1}{3+1} = \frac{1}{4} = \epsilon_0 \end{aligned}$$

Therefore, the convergence is not uniform.

4. (P.247 Q18) Note that the argument in Q8 actually implies the following: For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x \geq 0$, $\lim_{n\to\infty} xe^{-nx} = 0$. Therefore, the convergence is uniform.